2 Separable equations

We now start to systematically study *first order ODE* of the form

$$y' = f(x, y), \tag{1}$$

where f is a given function of two variables, x is an independent variable and $y = y(x)$ is the dependent variable or our unknown function. It is usually said that if our ODE has the form (1) then it is in *the normal form*. Do not expect to always see your problems written in the normal form.

Sometimes we will use the initial condition

$$y(x_0) = y_0, \quad x_0, y_0 \in \mathbf{R}, \tag{2}$$

and hence solve the IVP (1)–(2).

Most of the time in the first month of the course we will be learning how to solve problems of the form (1) or (1)–(2) *analytically*. This means that our goal is to actually figure out, manipulating somehow our equation, what is the function y that solves (1) — the answer should be a formula that depends on an arbitrary constant if we solve (1) and does not have any arbitrary constants if we solve (1)–(2). From the previous lecture we know how to accomplish this task for the ODE of the form

$$y' = f(x).$$

Here is an important generalization.

Definition 1. *An ODE of the form*

$$y' = f_1(x) f_2(y)$$

is called separable.

The analytical solution of the separable equation can be found by evaluating two integrals (well, not always, see below).

First, I present a mathematically rigorous approach. Start separating the variables:

$$\frac{y'}{f_2(y)} = f_1(x),$$

and integrate with respect to x both sides (remember that y actually depends on x)

$$\int \frac{y' \,\mathrm{d}x}{f_2(y)} = \int f_1(x) \,\mathrm{d}x.$$

Now, recalling from Calculus that $y'(x)\,\mathrm{d}x = \mathrm{d}y$ we get (technically, I make a change of variables in my left-hand side integral)

$$\int \frac{\mathrm{d}y}{f_2(y)} = \int f_1(x) \,\mathrm{d}x.$$

Now evaluate these two integrals, and we are done. Note that by evaluating two indefinite integrals we obtain two arbitrary constants, say, C_1 and C_2, however, if we move them to the same side and

MATH266: Intro to ODE by Artem Novozhilov, e-mail: artem.novozhilov@ndsu.edu. Spring 2024

use the notation $C := C_2 - C_1$ we end up with only one arbitrary constant, as should be expected for the first order equation.

There is another way of doing the same manipulations, that is based on the Leibnitz notations $y' = \frac{dy}{dx}$. Consider

$$\frac{dy}{dx} = f_1(x) f_2(y),$$

and collect now all the expressions containing y at the left hand side, and all the expressions with x at the right hand side:

$$\frac{dy}{f_2(y)} = f_1(x)\,dx.$$

Now by integrating we get exactly the same final answer

$$\int \frac{dy}{f_2(y)} = \int f_1(x)\,dx. \tag{3}$$

This formal manipulation usually will not get you to a wrong answer, however, it is useful to remember that $\frac{dy}{dx}$ is not a fraction, and hence we cannot simply treat dy and dx there as separate quantities.

Are we done with the separable equations? Not really, since we should be always worried when we divide something by an expression (in our case by $f_2(y)$), because we all know that we cannot divide by zero. Therefore, before formally dividing by $f_2(y)$ we should find out those values of y for which $f_2(y) = 0$.

Assume that $\hat{y}$ is such that $f_2(\hat{y}) = 0$. Now I claim that the function $y(x) = \hat{y}$ is also a solution to the separable equation. Indeed, left hand side of the equation is $y'(x) = (\hat{y})' = 0$ since $\hat{y}$ is a constants, the right-hand side is also zero, hence $y(x) = \hat{y}$ is a solution. Moreover, this kind of solutions can be lost if we simply stick to our formula (3).

Now to the examples.

Example 2 (The Malthus equation)**.** Recall from the previous lecture that the Malthus equation has the form

$$\dot{N} = mN, \quad m \in \mathbf{R},$$

where $N = N(t)$, and the dot denotes the derivative with respect to time t. This is a separable equation, because we can take, e.g., $f_1(t) = m$ and $f_2(N) = N$. Therefore, we can separate the variables. Before using (3), let us find for which N function $f_2(N)$ turns into zero. Obviously, it is true only for $\hat{N} = 0$. Therefore, we already found one solution to our equation: $N(t) = 0$. Now we can separate the variables and integrate

$$\int \frac{dN}{N} = \int m\,dt \implies \log|N| = mt + C_1.$$

Here log is the *natural logarithm*, which is also often denoted as $\ln := \log := \log_e$. Well, we are usually looking for a general solution in the form $N = N(t)$. It is not always possible to do so, but if it is possible, it should be done. Here we get

$$|N| = e^{mt+C_1} = C_2 e^{mt}, \quad C_2 := e^{C_1},$$

where I used a new arbitrary constant C_2, which now takes only positive values. Getting rid of the absolute value, we find

$$N = \pm C_2 e^{mt} = C_3 e^{mt}, \quad C_3 := \pm C_2,$$

where I again use a new arbitrary constant, which now takes any values from $\mathbf{R}$ except for zero. Finally, I recall that $N = 0$ actually is a solution, hence if instead of C_3 I take an arbitrary constant C, which takes any value from $\mathbf{R}$, then all possible solutions to the Malthus equation are given by

$$N(t) = Ce^{mt}.$$

This formula is the general solution to the Malthus equation. It is quite important, so you should memorize it. *Q:* Can you sketch the graphs of $N(t)$ for positive and negative m?

Carefully look through the solution process for this equation. In most of the cases we will not be spending our time to painstakingly distinguish all the arbitrary constants in the calculations. With more experience, the shortcut solution in the form

$$\dot{N} = mN \implies \int \frac{\mathrm{d}N}{N} = \int m \,\mathrm{d}t \implies \log|N| = mt + C \implies N(t) = Ce^{mt}$$

will be sufficient, especially if you understand all the hidden steps in this derivation. Note also that I used the same letter C to denote different arbitrary constants. This will be the rule: any arbitrary constant will be usually denoted as C, and one C can be different from C on another line.

Finally, convince yourself that if there is the initial condition $N(t_0) = N_0$, then the particular solution is

$$N(t) = N_0 e^{m(t-t_0)}.$$

Example 3. We are given

$$y' = y^2 t.$$

Separate the variables and integrate

$$\begin{aligned} \int \frac{\mathrm{d}y}{y^2} &= \int t \,\mathrm{d}t \implies \\ -\frac{1}{y} &= \frac{t^2}{2} + C \implies \\ \frac{1}{y} &= C - \frac{t^2}{2} \implies \\ y &= \frac{1}{C - \frac{t^2}{2}} \implies \\ y &= \frac{2}{C - t^2}. \end{aligned}$$

Please make sure that you understand how the arbitrary constants, all of which are denoted by the same letter C, are related to each other.

Did we find the general solution? Not really, because, as you can check, $y = 0$ is also a solution to our equation (plug it in). Convince yourself that for no value of C our formula gives us this particular solution. Hence, the final answer is

$$y(t) = \frac{2}{C - t^2} \quad \text{or} \quad y(t) = 0.$$

Example 4 (Logistic equation[1]). The Malthus equation is unrealistic in the sense that it predicts unlimited exponential growth in the case $m > 0$. On the other hand, we know that no population can grow to infinity. Here is how we can fix this issue. While solving Malthus equation, it was supposed that m is a constant. It is more realistic to assume that $m = m(N)$, i.e., a function that depends on the current population size. What is the simplest function different from a constant? A reasonable answer would be "a linear function $m(N) = a + bN$," hence we get

$$\dot{N} = (a + bN)N.$$

More often a different parametrization is used:

$$\dot{N} = rN\left(1 - \frac{N}{K}\right),$$

and this ODE is often called the *logistic equation* (*Q:* Can you figure out how a, b are related to r, K?). Both constants here are supposed to be positive, and K is called *the carrying capacity* because its value predicts the final population size (see below).

This is a separable equation, let us solve it (again, skipping some of the steps):

$$\begin{aligned}
\dot{N} &= rN\frac{(K-N)}{K} \implies \\
\int \frac{K\,\mathrm{d}N}{N(K-N)} &= \int r\,\mathrm{d}t \implies \\
\int \left(\frac{1}{N} + \frac{1}{K-N}\right)\mathrm{d}N &= \int r\,\mathrm{d}t \implies \\
\log|N| - \log|K-N| &= rt + C \implies \\
\frac{N}{K-N} &= Ce^{rt} \implies \\
N(t) &= \frac{KCe^{rt}}{1+Ce^{rt}} \implies \\
N(t) &= \frac{K}{1+Ce^{-rt}}.
\end{aligned}$$

Again, C in one line is probably different from C in another line. Did we find all the solutions? Using the reasoning described above, we note that there are two values of N for which the right hand side of the logistic equation vanishes: $\hat{N}_1 = 0$ and $\hat{N}_2 = K$. Hence we also have the solutions $N(t) = 0$ and $N(t) = K$. The second solution can be obtained if we put $C = 0$ in our solution found by integration, and $N = 0$ cannot be obtained for any value of C. Hence the general solution to the logistic equation is

$$N(t) = \frac{K}{1+Ce^{-rt}} \quad \text{or} \quad N(t) = 0.$$

If we were given the initial condition $N(0) = N_0$, we would get $C = \frac{K-N_0}{N_0}$, and (check!)

$$N(t) = \frac{KN_0}{N_0 + (K-N_0)e^{-rt}}.$$

[1] If you have some time and like reading about history of some important equations, I encourage you to find online and read Kingsland, S. (1982). The refractory model: The logistic curve and the history of population ecology. The Quarterly Review of Biology, 57(1), 29–52.

Note that if $N_0 > 0$ then $N(t) \to K$ when $t \to \infty$, there is no unlimited growth.

We will return to the logistic equation in the forthcoming lectures.

Example 5 (The law of radioactive decay and radiocarbon dating)**.** To conclude this section I would like to demonstrate that although we just started learning the differential equations, the knowledge we have can be used to obtain very interesting and important conclusions about the world around us. Specifically, I would like to explain how the so-called *radiocarbon dating method* can be used to determine the age of ancient organic fossils.

The basic law of radioactive decay, which was an experimentally observed fact, states that the ratio of the number of atoms that disintegrate during time unit to the total number of atoms equals to a constant that is determined by the nature of the material and not by its amount. Let $N(t)$ be the number of the atoms at time t, hence during the time interval $\mathrm{d}t$ the number that have been disintegrated is $N(t) - N(t + \mathrm{d}t)$, therefore the law that I just stated can be cast in the form of the following formula:

$$\frac{N(t) - N(t + \mathrm{d}t)}{N(t)} \approx k\,\mathrm{d}t,$$

where k is exactly the constant that I mentioned before, which is often, somewhat incorrectly, is called *the probability of disintegration*. Rearranging the terms and taking the limit $\mathrm{d}t \to 0$ yields (fill in the omitted steps) the ODE

$$\dot{N} = -kN,$$

with the initial condition $N(t_0) = N_0$. Please note that this is not a new equation, this is exactly the Malthus equation that we already solved! The solution is given by

$$N(t) = N_0 e^{-k(t-t_0)}\,.$$

In practice it is important to measure constant k for different materials. This can be done using the observation that the time τ, which is required for the given material to decrease exactly in half, does not depend on the initial condition N_0. Indeed, to find this time I must solve (assuming without the loss of generality that $t_0 = 0$)

$$\frac{N_0}{2} = N_0 e^{-k\tau}\,,$$

which implies that

$$k = \frac{\log 2}{\tau}\,,$$

and if we know τ hence we know k. It turns out that τ, which for the obvious reasons is called *the half-life*, can me measured experimentally. For instance it is known that the half-life of carbon-14, which is denoted ^{14}C, is approximately 5730 years.

Finally I get to the idea how the radiocarbon dating works. When, e.g., a tree (or some other organic object) is alive, the amount of carbon-14 in it is at equilibrium: The amount absorbed from the atmosphere is equal to the amount radiated (disintegrated). As soon as the tree dies, it keeps disintegrating carbon-14 but does not absorb it any longer. If N_0 is the number of atoms of ^{14}C at the moment the tree died (this number is equal to the observed one in the biosphere at that time) and $N(t)$ is the number of atoms of ^{14}C in the sample of this dead tree (which can be measured) then, using the formulas above, the age of the tree can be estimated as[2]

$$t = \frac{1}{k} \log \frac{N_0}{N(t)} = \tau \log_2 \frac{N_0}{N(t)} .$$

[2]The real life is certainly more complicated.